Higher-order corrections & IR divergences

We'll now compute higher-order perturbative corrections to the R-ratio. We'll find that these corrections turn out to be numerically small. However, we'll encounter some interesting complications along the way, namely that the diagrams suffer from infrared (IR) divergences. Understanding their structure and cancellation helps us to design other, less inclusive variables. It also guides the construction

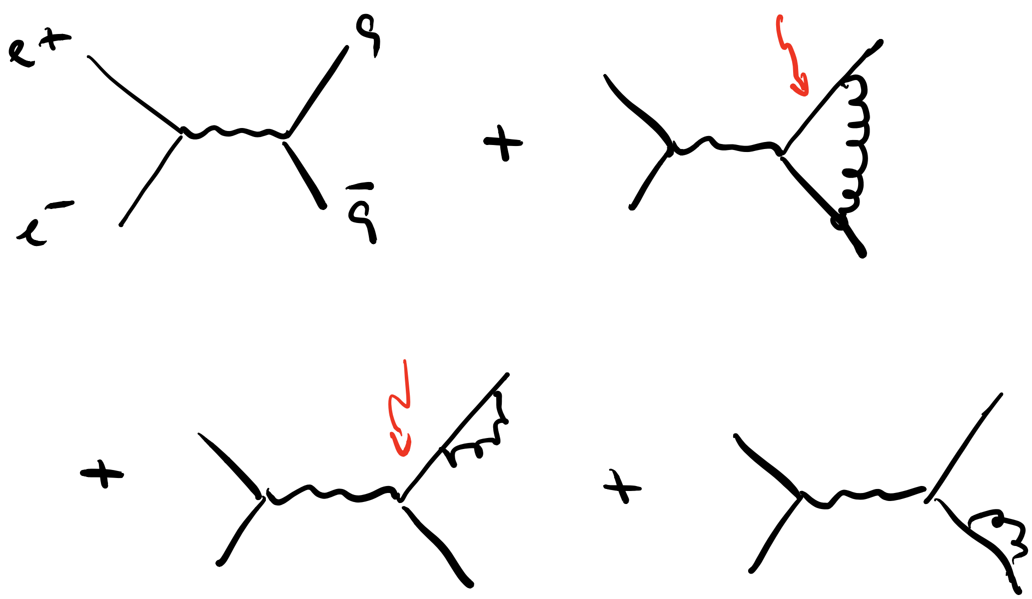
Compute R-Ratio to NLO; $O(\alpha_s)$

$$\sigma(e^+e^- \to X)$$

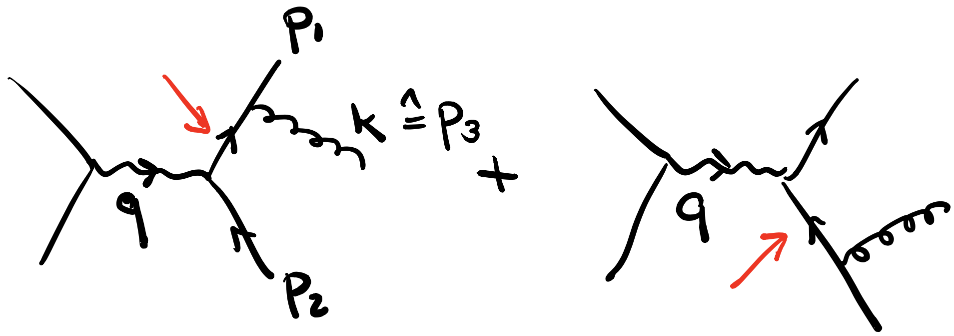
$$= \sigma(e^+e^- \to \bar{q}q)^{A}$$

$$+ \sigma(e^+e^- \to q\bar{q}g)^{B} + O(\alpha_s^2)$$

A.)



B.)



$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{16\pi}{Q^2}\, \sigma^{(0)}_{q\bar{q}}\, C_F \; \frac{(p_1\cdot k)^2 + (p_2\cdot k)^2 + q^2\, p_1\cdot p_2}{p_1\cdot k \;\; p_2\cdot k}$$

Amplitude diverges if $p_1 \parallel k$, $p_2 \parallel k$

or if $k \to 0$.

Parameterize:

$$(p_1+p_2)^2 = y_3 q^2$$

$$(p_1+k)^2 = y_2 q^2 = 2p_1\cdot k$$

$$(p_2+k)^2 = y_1 q^2$$

In the CMS system, we have

$$q^\mu = (Q, \vec{0}) = p_1^\mu + p_2^\mu + k^\mu$$

$$y_i = 1 - \frac{2E_i}{Q} = 0 \dots 1 \qquad Q = \sqrt{q^2}$$

Note:

$$(p_1+p_2)^2 = (q-k)^2 = Q^2 - 2Q \overset{k_0}{\overset{\downarrow}{E_3}}$$

$$= Q^2 \left(1 - \frac{2E_3}{Q}\right)$$

$$= Q^2 y_3$$

$$y_1 + y_2 + y_3 = 3 - \frac{2(\overbrace{E_1+E_2+E_3}^{Q})}{Q}$$

$$= 3 - 2 = 1$$

$$\int d\Pi_3 = \frac{q^2}{128\pi^3} \int_0^1 dy_1 \int_0^{1-y_1} dy_2$$

cross section becomes

$$\sigma_{q\bar{q}g} = \sigma^{(0)}_{q\bar{q}} \frac{\alpha_s C_F}{\pi} \int_0^1 dy_1 \int_0^{1-y_1} dy_2 \; \frac{4y_3 + 2(y_1^2 + y_2^2)}{y_1 y_2}$$

divergent integral!

Perform the entire computation in $d = 4 - 2\varepsilon$. Use ε to regularize IR divergencies.

$$PS_3 = \int \frac{d^{d-1} p_1}{2E_1 (2\pi)^{d-1}} \int \frac{d^{d-1} p_2}{2E_2 (2\pi)^{d-1}} \int \frac{d^{d-1} k}{2E_3 (2\pi)^{d-1}}$$

$$(2\pi)^d \delta^{(d)}(q - p_1 - p_2 - k)$$

We can immediately integrate over k to get

$$PS_3 = \frac{1}{8(2\pi)^{2d-3}} \int \frac{d^{d-1} p_1}{E_1} \int \frac{d^{d-1} p_2}{E_2}$$

$$\times \; \delta(Q - E_1 - E_2 - E_3)$$

By momentum conservation $\vec{k} = -\vec{p}_1 - \vec{p}_2$

and $E_i = |\vec{p}_i|$ since the particles are all massless. We thus have

angle between $\vec{p}_1$ & $\vec{p}_2$

$$E_3^2 = (\vec{p}_1 + \vec{p}_2)^2 = E_1^2 + E_2^2 + 2E_1E_2\cos\theta .$$

By spherical symmetry, we can choose

$$p_1^\mu = E_1(1, \overbrace{1, 0, \dots}^{(d-1)})$$

For p_2^μ we choose

unit vector in $d-2$.

$$p_2^\mu = E_2(1, \cos\theta, \sin\theta\, \vec{n}_{d-2})$$

We then have

$$PS_3 = \frac{1}{8(2\pi)^{2d-3}} \int_0^{Q/2} dE_1\, E_1^{d-3} \int d\Omega_{d-1}$$

$$\cdot \int_0^{Q/2-E_1} dE_2\, E_2^{d-3} \int_0^{\pi} d\theta\, |\sin\theta|^{d-3} \int d\Omega_{d-2}$$

$$\cdot \frac{1}{E_3}\,\delta(Q - E_1 - E_2 - E_3)$$

Now set $E_1 = (1-y_1)Q/2$; $E_2 = (1-y_2)Q/2$

and integrate over θ.

Note:

$$\int d\Omega^{d-1} = \int d^{d-1}\vec{n}\;\delta(1-\vec{n}^2) =$$

$$\int_{-1}^{1} d\cos\theta\,|\sin\theta|^{d-2}\int d^{d-2}\vec{n}\;\delta(\underbrace{1-\cos^2\theta}_{\sin^2\theta} - \sin^2\theta\,\vec{n}^2_{d-2})$$

$$= \int_{-1}^{1} d\cos\theta\,|\sin\theta|^{d-4}\int d^{d-2}\vec{n}\;\delta(1-\vec{n}^2_{d-2})$$

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

see the notebook R_ratio_NLO.nb

for a step-by-step computation

of the three-particle phase space.

The result is

$$PS_3 = \frac{Q^2}{128\pi^3}\left(\frac{4\pi}{Q^2}\right)^{2\varepsilon}\frac{1}{\Gamma(2-2\varepsilon)}\int_0^1 dy_1 \int_0^{1-y_1} dy_2$$

$$(y_1 y_2 y_3)^{-\varepsilon} \quad \text{with } y_3 = 1-y_2-y_1$$

To get the full d-dim result, we also

need to recompute $|M|^2$ in d-dimensions,

since the fermion traces yield factors

of ε.

Combining the amplitude with PS_3 and dividing by $\sigma_{q\bar{q}}^{(0)}$ (in d-dim!), we get

$$\sigma_{q\bar{q}g} = \sigma_{q\bar{q}}^{(0)} \cdot \frac{\alpha_s C_F}{4\pi \Gamma(1-\varepsilon)} \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\varepsilon}$$

$$\cdot \int_0^1 dy_1 \int_0^{1-y_1} dy_2 \, (y_1 y_2 y_3)^{-\varepsilon}$$

$$\times \frac{4(y_3 - y_1 y_2 \varepsilon) + 2(1-\varepsilon)(y_1^2 + y_2^2)}{y_1 y_2}.$$

The next step is to perform the y-integrations. Simply performing them in d-dim is not easily possible, but one can also not just expand the integrand, because of the divergencies.

The integrals have the form:

$$\int_0^1 dy\, y^{-1-\varepsilon} f(y) = \int_0^1 dy\, y^{-1-\varepsilon} \big(f(y) - f(0) \big) + \int_0^1 dy\, y^{-1-\varepsilon} f(0)$$

regular function

$$= -\frac{1}{\varepsilon} f(0) + \int_0^1 dy\, \underbrace{y^{-1-\varepsilon} \left[f(y) - f(0) \right]}_{\text{convergent, expand in } \varepsilon \text{ before integration}}$$

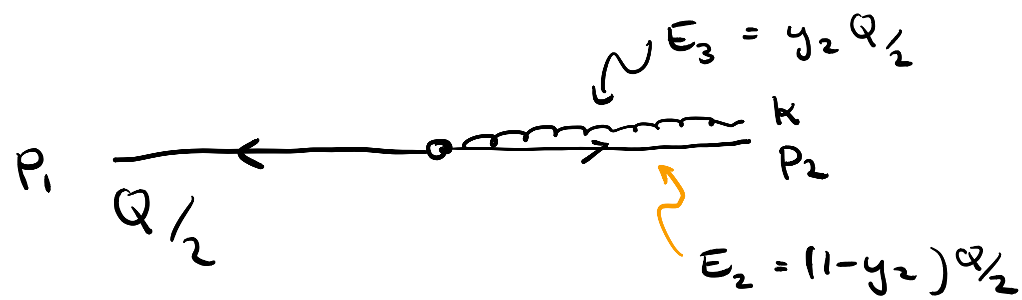
A bit more elegantly, one can write

$$y^{-1+\alpha} = \frac{1}{\alpha} \delta(y) + \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left(\frac{\ln^n(y)}{y} \right)_+$$

(exercise)

See the Notebook for the explicit computation.

It is interesting to explore, where the singularities are coming from: if we go to $y_1 = 0$, then $E_1 = Q/2$. This is only possible if the other particles fly collinearly in the other direction:



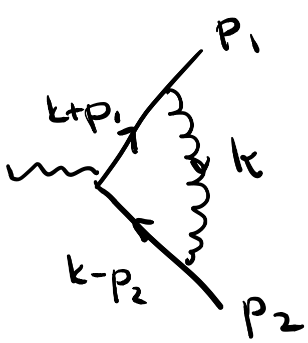
In this situation y_2 corresponds to the momentum fraction carried by the collinear gluon. A physically well motivated way to compute the phase-space integral is to analyze it's soft & collinear limits.

The final result reads

$$\sigma_{q\bar{q}g} = \sigma_{q\bar{q}}^{(0)} \frac{\alpha_s C_F}{4\pi} \left(\frac{4\pi e^{\gamma_E} \mu^2}{Q^2} \right)^{\varepsilon}$$

$$\left\{ \frac{4}{\varepsilon^2} + \frac{6}{\varepsilon} + 19 - \frac{7\pi^2}{3} \right\}$$

Next, we need to compute the virtual corrections $\mathcal{M}_{q\bar{q}}^{(1)}$. We need



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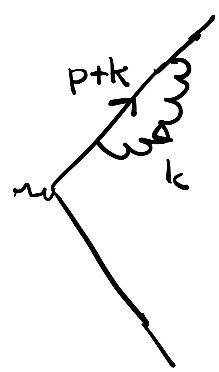
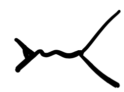
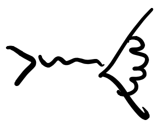


To extract Z-factor

↳ $\frac{Z_{OS}}{\not{p}}$

$\propto (p^2)^{-\varepsilon} = 0$ for $p^2 = 0$!

Note :

$$\sigma_{q\bar{q}} = \Bigg|$$  $$+$$  $$\Bigg|^2$$

$$= |\mathcal{M}^{(0)} + \mathcal{M}^{(1)}|^2$$

$$= |\mathcal{M}^{(0)}|^2 + \mathcal{M}^{(0)}\mathcal{M}^{(1)*} + \mathcal{M}^{(1)}\underbrace{\mathcal{M}^{(0)*}}_{} \qquad = \mathcal{M}^{(0)}$$

$$+ \mathcal{O}(\alpha_s^2)$$

$$= |\mathcal{M}^{(0)}|^2 + 2\mathcal{M}^{(0)}\,\mathrm{Re}(\mathcal{M}^{(1)}).$$

$$= \sigma^{(0)}_{q\bar{q}} \left\{ 1 + \frac{\alpha_s C_F}{4\pi}\left(\frac{4\pi e^{-\gamma_E}\mu^2}{Q^2}\right)^{\varepsilon} \left[-\frac{4}{\varepsilon^2} - \frac{6}{\varepsilon} - 16 + \frac{7\pi^2}{3} \right] \right\}$$

Now we can finally add up

$$\sigma(e^+e^- \to X) = \sigma_{q\bar{q}} + \sigma_{q\bar{q}g} + \mathcal{O}(\alpha_s^2)$$

$$= \sigma_{q\bar{q}}^{(0)} \left\{ 1 + \underbrace{3 C_F \frac{\alpha_s}{4\pi}}_{3 \cdot \frac{4}{3} \cdot \frac{0.1}{12} \approx 0{,}03} \right\}$$

A very small $\sim 3\%$, but finite result.